PHYS 101 Midterm Exam 2 Solutions 16.04.2022

1. Potential energy function of a conservative force is given by the expression $U(x) = k_1 x + k_2/x$, $(x > 0)$, where k_1 and k_2 are positive constants, and x is in meters.

(a) (4 Pts.) What are the units of the constants k_1 and k_2 in terms of the basic units kg, m, s?

(b) (6 Pts.) What is the force (vector) as a function of x?

(c) (5 Pts.) Find the point of equilibrium. Is it stable or unstable? Why?

(d) (10 Pts.) How much work is done by this conservative force if an object is displaced from $x = 1$ m to the point $x = 2$ m?

Solution:

(a) $[k_1] = \mathrm{kg\,.\,m/s^2}$, $[k_2] = \mathrm{kg\,.\,m^3/s^2}$

(b) $F_x = -\left(\frac{dU}{dx}\right) \quad \rightarrow \quad \vec{\mathbf{F}} = \left(-k_1 + \frac{k_2}{x^2}\right)\,\hat{\mathbf{i}}\,.$

(c) $\vec{\mathbf{F}} = 0 \quad \rightarrow \quad x_{\mathrm{eq}} = \sqrt{\frac{k_2}{k_1}}\,, \qquad x_{\mathrm{eq}} > 0\,.$

Equilibrium is stable because

$\frac{d^2U}{dx^2} = 2\frac{k_2}{x^3}$, which is positive at $x = x_{\mathrm{eq}} = \sqrt{\frac{k_2}{k_1}}$.

(d) $W_F = -\Delta U$

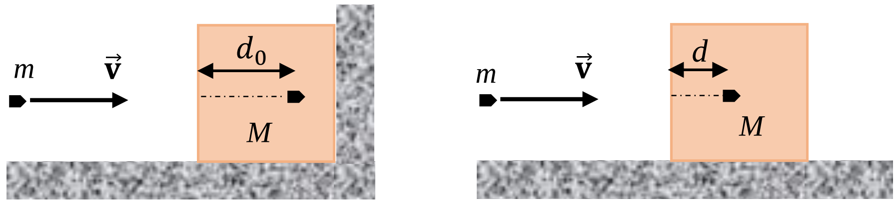
$U(x=2) = 2k_1 + k_2/2\,, \qquad U(x=1) = k_1 + k_2\,, \qquad \Delta U = k_1 - k_2/2\,.$

Therefore,

$W_F = -k_1 + \frac{k_2}{2}\,.$

2. A plastic block with mass M is on a frictionless horizontal surface resting against a wall, as shown in the figure on the left below. A bullet with mass m is shot into it with speed v, and it is observed that the block does not move as the bullet penetrates a distance d_0 into the block before stopping in a negligibly short time.

(a) (5 Pts.) What is the magnitude of the impulse delivered to the bullet by the block?



(b) (20 Pts.) Assuming that the same constant friction force between the plastic and the bullet stops the bullet in both cases, find the distance d the bullet would penetrate in term of M, m and d_0, if there had been no wall to rest against (figure on the right above).

Solution:

(a) $\vec{\mathbf{J}} = \Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i \quad \rightarrow \quad |\vec{\mathbf{J}}| = m|\vec{\mathbf{v}}| = mv\,.$

(b) When the block is resting against the wall, it is the constant friction force which stops the bullet. We can find the friction force f by using the work - energy theorem.

$$\Delta K = W_f \quad \rightarrow \quad -\frac{1}{2}mv^2 = -fd_0 \quad \rightarrow \quad f = \frac{mv^2}{2d_0}\,.$$

When the block is not resting against the wall, collision between the bullet and the block is completely inelastic. Velocity of the block + the bullet after the collision is found by using the fact that momentum of the system is conserved.

$$p_{xi} = mv\,, \qquad p_{xf} = (m+M)v' \quad \rightarrow \quad v' = \frac{mv}{m+M}\,.$$

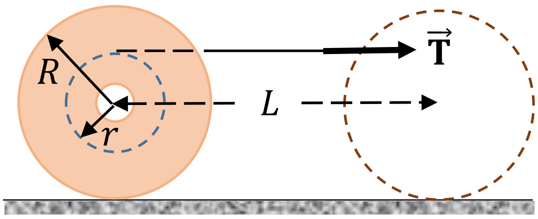
Amount of kinetic energy lost in the collision is

$$\Delta K = \frac{1}{2}(m+M)\left(\frac{mv}{m+M}\right)^2 - \frac{1}{2}mv^2 = -\frac{1}{2}\left(\frac{mM}{m+M}\right)v^2\,.$$

According to the work – energy theorem

$$\Delta K = W_f \quad \rightarrow \quad -\frac{1}{2}\left(\frac{M}{m+M}\right)mv^2 = -fd = -\left(\frac{mv^2}{2d_0}\right)d \quad \rightarrow \quad d = \frac{Md_0}{m+M}\,.$$

3. A cylindrically symmetric spool of mass M, inner radius r, outer radius R, and moment of inertia I is initially at rest on a horizontal surface. Starting at time $t = 0$, a tension of magnitude T is applied to the free end of the massless string wound around the inner cylinder, as shown in the figure. The spool rolls without slipping a distance L along the horizontal surface.

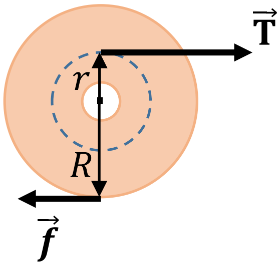


(a) (10 Pts.) What is the speed of the center of the spool after rolling the distance L?

(b) (10 Pts.) What is the kinetic energy of the spool after rolling the distance L?

(c) (5 Pts.) How much string is unwound during the motion?

Solution: (a) Free-body diagram of the spool is given on the right. Since the spool rolls without slipping, relation between its angular acceleration α and the linear acceleration of its center is $a_c = R\alpha$. Writing Newton's second law, we find the linear acceleration of the center as



$$T - f = Ma_c\,, \qquad rT + Rf = I\alpha \quad \rightarrow \quad a_c = \left(\frac{R^2 + rR}{I + MR^2}\right)T\,.$$

The speed of the center of the spool after rolling the distance L can be found by using the kinematical relation

$$v^2 - v_0^2 = 2a_c L \quad \rightarrow \quad v^2 = 2\left(\frac{R^2 + rR}{I + MR^2}\right)TL \quad \rightarrow \quad v = \sqrt{2\left(\frac{R^2 + rR}{I + MR^2}\right)TL}\,.$$

(b) Kinetic energy of the spool after rolling the distance L is

$$K = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}\left(M + \frac{I}{R^2}\right)v^2 \quad \rightarrow \quad K = \left(1 + \frac{r}{R}\right)TL\,.$$

(c) With each rotation the center of the spool moves a distance $2\pi R$. Hence, to cover the distance L the spool makes $L/(2\pi R)$ rotations. Amount of string unwound with each rotation is $2\pi r$. Therefore, total amount of string unwound is

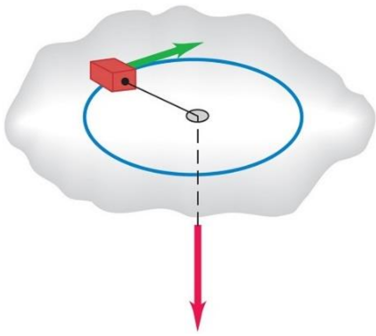
$$\Delta s = (2\pi r)\left(\frac{L}{2\pi R}\right) \quad \rightarrow \quad \Delta s = \left(\frac{r}{R}\right)L\,.$$

Since the spool also moves a distance L, the total distance over which the tension T acts on the spool is $d = L + rL/R$. Work done by the tension is found as

$$W_T = \left(1 + \frac{r}{R}\right)LT\,,$$

in accordance with the work energy theorem.

4. A small block with mass m is on a frictionless horizontal surface. It is attached to a massless cord passing through a hole in the surface, as shown in the figure. The block is initially rotating at a distance R_1 from the hole with angular speed ω_1. The cord is pulled from below, shortening the radius of the circle in which the block revolves to R_2. Model the block as a particle.



(a) (10 Pts.) What is the new angular speed?

(b) (10 Pts.) Find the change in the kinetic energy of the block.

(c) (5 Pts.) How much work was done in pulling the cord?

Solution:

(a) The force shortening the string is in the radial direction. Therefore, the net torque on the rotating block is zero. This means that the angular momentum of the block is conserved. Treating the block as a particle, we have

$$L_1 = I_1\omega_1 = mR_1^2\omega_1\,, \qquad L_2 = I_2\omega_2 = mR_2^2\omega_2\,, \qquad L_1 = L_2 \quad \rightarrow \quad \omega_2 = \left(\frac{R_1}{R_2}\right)^2 \omega_1\,.$$

(b) $K_1 = \frac{1}{2}mR_1^2\omega_1^2\,, \qquad K_2 = \frac{1}{2}mR_2^2\omega_2^2 = \frac{1}{2}mR_2^2\left(\frac{R_1}{R_2}\right)^4 \omega_1^2 \quad \rightarrow \quad K_2 = \frac{mR_1^4}{2R_2^2}\omega_1^2\,.$

So, $\qquad \Delta K = \left(\frac{R_1^2}{R_2^2} - 1\right)\frac{1}{2}mR_1^2\omega_1^2\,.$

(c) By the work – energy theorem, work done in pulling the chord is equal to the change in the kinetic energy of the block.

$$W = K = \left(\frac{R_1^2}{R_2^2} - 1\right)\frac{1}{2}mR_1^2\omega_1^2\,.$$